

Cosmological consequences of the noncommutative spectral geometry as an approach to unification

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Abstract. Noncommutative spectral geometry succeeds in explaining the physics of the Standard Model of electroweak and strong interactions in all its details as determined by experimental data. Moreover, by construction the theory lives at very high energy scales, offering a natural framework to address early universe cosmological issues. After introducing the main elements of noncommutative spectral geometry, I will summarise some of its cosmological consequences and discuss constraints on the gravitational sector of the theory.

1. Introduction

Approaching Planckian energies, gravity can no longer be considered as a classical theory, the quantum nature of space-time becomes apparent and geometry can no longer be described in terms of Riemannian geometry and General Relativity. At such energy scales, all forces should be unified so that all interactions correspond to one underlying symmetry. The nature of space-time would change in such a way, so that its low energy limit implies the diffeomorphism and internal gauge symmetries, which govern General Relativity and gauge groups on which the Standard Model is based, respectively. Thus, near Planckian energies one should search for a formulation of geometry within the quantum framework. Such an attempt has been realised within NonCommutative Geometry (NCG) [1, 2]. Even though this approach is still in terms of an effective theory, it has already led to very encouraging results and is now at a stage to be confronted with experimental and observational data.

Since all physical data are of a spectral nature, we will use the notion of a spectral triple, which is analogous to Fourier transform we are familiar with in commutative spaces. Distances are measured in units of wavelength of atomic spectra, while the notion of a real variable, taken as a function on a set X , will be replaced by a self-adjoint operator D in a Hilbert space. The space X will be described by the algebra \mathcal{A} of coordinates, represented as operators in a fixed Hilbert space \mathcal{H} . Thus, the usual emphasis on the points of a geometric space is replaced by the spectrum of an operator and the geometry of a noncommutative space is determined in terms of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Noncommutative geometry spectral action, in its current version which is still at the classical level and it certainly considers almost commutative spaces, it nevertheless offers [3] an elegant geometric interpretation of the most successful phenomenological model of particle physics, namely the Standard Model (SM). Certainly if the Large Hadron Collider (LHC) supports evidence for physics beyond the SM, one should consider extensions of the effective theory

we will adopt here. Closer to the Planck era one should consider the full, still unknown, theory, while at even higher energy scales the whole concept of geometry may lose its familiar meaning. Nevertheless, the effective theory we will consider here lives, by construction, at very high energies, offering an excellent framework to address open questions of early universe cosmology [4].

In what follows, I will briefly review the main elements of the NCG spectral action as an approach to unification and then discuss some of its cosmological consequences [4, 5, 6, 7, 8, 9].

2. NCG spectral action

In the context of noncommutative geometry, all information about a physical system is contained within the algebra of functions, represented as operators in a Hilbert space, while the action and metric properties are encoded in a generalised Dirac operator. The geometry is specified by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, defined by an algebra \mathcal{A} , a Hilbert space \mathcal{H} and a generalised Dirac operator D .

Adopting the simplest generalisation beyond commutative spaces, we take the extension of our smooth four-dimensional manifold, \mathcal{M} , by taking its product with a discrete noncommuting manifold \mathcal{F} . Thus, \mathcal{M} describes the geometry for the four-dimensional space-time, while the noncommutative space \mathcal{F} specifies the internal geometry for the SM and is composed of just two points. Considering the Standard Model of electroweak and strong interactions as a phenomenological model, we will look for a geometry, such that the associated action functional leads to the SM with all specifications as determined by experimental data.

Within NCG, a space described by the algebra of real coordinates is represented by self-adjoint operators on a Hilbert space. Since real coordinates are represented by self-adjoint operators, all information about a space is encoded in the algebra of coordinates \mathcal{A} , which is the main input of the theory. Under the assumption that the algebra constructed in $\mathcal{M} \times \mathcal{F}$ is symplectic-unitary, it turns out that \mathcal{A} must be of the form

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}), \quad (1)$$

with $k = 2a$ and \mathbb{H} denoting the algebra of quaternions. The choice $k = 4$ is the first value that produces the correct number of fermions, namely $k^2 = 16$, in each of the three generations [10]. The number of generations is a physical input. Certainly if at LHC new particles are discovered, one may be able to accommodate them by including a higher value for the even number k .

The operator D corresponds to the inverse of the Euclidean propagator of fermions, and is given by the Yukawa coupling matrix which encodes the masses of the elementary fermions and the Kobayashi–Maskawa mixing parameters. The fermions of the SM provide the Hilbert space \mathcal{H} of a spectral triple for the algebra \mathcal{A} , while the bosons of the SM are obtained through inner fluctuations of the Dirac operator of the product $\mathcal{M} \times \mathcal{F}$ geometry.

The spectral action principle states that the bosonic part of the spectral functional S depends only on the spectrum of the Dirac operator and its asymptotic expression, and for large energy Λ is of the form $\text{Tr}(f(D/\Lambda))$, with f being a cut-off function, whose choice plays only a small rôle. The physical Lagrangian has also a fermionic part, which has the simple linear form $(1/2)\langle J\psi, D\psi \rangle$, where J is the real structure on the spectral triple and ψ are spinors defined on the Hilbert space [11]. Applying the spectral action principle to the inner fluctuations of the product geometry $\mathcal{M} \times \mathcal{F}$, one recovers the Standard Model action coupled to Einstein and Weyl gravity plus higher order nonrenormalisable interactions suppressed by powers of the inverse of the mass scale of the theory [3].

To study the implications of this noncommutative approach coupled to gravity for the cosmological models of the early universe, one can concentrate just on the bosonic part of the action; the fermionic part is however crucial for the particle physics phenomenology of the model.

Using the heat kernel method, the bosonic part of the spectral action can be expanded in powers of the scale Λ in the form [3, 12, 13]

$$\text{Tr} \left(f \left(\frac{D}{\Lambda} \right) \right) \sim \sum_{k \in \text{DimSp}} f_k \Lambda^k \int |D|^{-k} + f(0) \zeta_D(0) + \mathcal{O}(1) , \quad (2)$$

with the momenta f_k of the cut-off function f given by

$$f_k \equiv \int_0^\infty f(u) u^{k-1} du \quad \text{for } k > 0 , \quad \text{and} \quad f_0 \equiv f(0) ; \quad (3)$$

the noncommutative integration is defined in terms of residues of zeta functions, $\zeta_D(s) = \text{Tr}(|D|^{-s})$ at poles of the zeta function, and the sum is over points in the *dimension spectrum* of the spectral triple.

In this way, one obtains a Lagrangian which contains in addition to the full SM Lagrangian, the Einstein-Hilbert action with a cosmological term, a topological term related to the Euler characteristic of the space-time manifold, a conformal Weyl term and a conformal coupling of the Higgs field to gravity. Writing the asymptotic expansion of the spectral action, a number of geometric parameters appear, which describe the possible choices of Dirac operators on the finite noncommutative space. These parameters correspond to the Yukawa parameters of the particle physics model and the Majorana terms for the right-handed neutrinos. The Yukawa parameters run with the Renormalisation Group Equations (RGE) of the particle physics model. Since running towards lower energies, implies that nonperturbative effects in the spectral action cannot be any longer neglected, any results based on the asymptotic expansion and on renormalisation group analysis can only hold for early universe cosmology. For later times, one should instead consider the full spectral action.

More precisely, the bosonic action in Euclidean signature reads [3]

$$\begin{aligned} \mathcal{S}^E = \int & \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^* R^* + \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu i} + \frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} \right. \\ & \left. + \frac{1}{4} B^{\mu\nu} B_{\mu\nu} + \frac{1}{2} |D_\mu \mathbf{H}|^2 - \mu_0^2 |\mathbf{H}|^2 - \xi_0 R |\mathbf{H}|^2 + \lambda_0 |\mathbf{H}|^4 \right) \sqrt{g} d^4 x , \end{aligned} \quad (4)$$

where

$$\begin{aligned} \kappa_0^2 &= \frac{12\pi^2}{96f_2\Lambda^2 - f_0\mathfrak{c}} , \\ \alpha_0 &= -\frac{3f_0}{10\pi^2} , \\ \gamma_0 &= \frac{1}{\pi^2} \left(48f_4\Lambda^4 - f_2\Lambda^2\mathfrak{c} + \frac{f_0}{4}\mathfrak{d} \right) , \\ \tau_0 &= \frac{11f_0}{60\pi^2} , \\ \mu_0^2 &= 2\Lambda^2 \frac{f_2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} , \\ \xi_0 &= \frac{1}{12} , \\ \lambda_0 &= \frac{\pi^2\mathfrak{b}}{2f_0\mathfrak{a}^2} , \end{aligned} \quad (5)$$

with $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ given by [3]

$$\begin{aligned}
\mathfrak{a} &= \text{Tr} \left(Y_{(\uparrow 1)}^* Y_{(\uparrow 1)} + Y_{(\downarrow 1)}^* Y_{(\downarrow 1)} + 3 \left(Y_{(\uparrow 3)}^* Y_{(\uparrow 3)} + Y_{(\downarrow 3)}^* Y_{(\downarrow 3)} \right) \right) , \\
\mathfrak{b} &= \text{Tr} \left(\left(Y_{(\uparrow 1)}^* Y_{(\uparrow 1)} \right)^2 + \left(Y_{(\downarrow 1)}^* Y_{(\downarrow 1)} \right)^2 + 3 \left(Y_{(\uparrow 3)}^* Y_{(\uparrow 3)} \right)^2 + 3 \left(Y_{(\downarrow 3)}^* Y_{(\downarrow 3)} \right)^2 \right) , \\
\mathfrak{c} &= \text{Tr} (Y_R^* Y_R) , \\
\mathfrak{d} &= \text{Tr} \left((Y_R^* Y_R)^2 \right) , \\
\mathfrak{e} &= \text{Tr} \left(Y_R^* Y_R Y_{(\uparrow 1)}^* Y_{(\uparrow 1)} \right) ,
\end{aligned} \tag{6}$$

with $Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_{(\downarrow 3)}, Y_{(\uparrow 3)}$ and Y_R being (3×3) matrices, with Y_R symmetric; the Y matrices are used to classify the action of the Dirac operator and give the fermion and lepton masses, as well as lepton mixing, in the asymptotic version of the spectral action. Note that \mathbf{H} is a rescaling of the Higgs field, so that the kinetic terms are normalised. One should be cautious that the relations in Eq. (5) are only valid at unification scale Λ ; it is incorrect to consider them as functions of the energy scale.

The noncommutative spectral geometry approach leads to various phenomenological consequences. Normalisation of the kinetic terms implies

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4} \quad \text{and} \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2 ,$$

while

$$\sin^2 \theta_W = \frac{3}{8} ; \tag{7}$$

a relation which holds also for SU(5) and SO(10), while assuming the *big desert* hypothesis, one can find the running of the three couplings $\alpha_i = g_i^2/(4\pi)$. One-loop RGE for the running of the gauge couplings and the Newton constant, shows that they do not meet exactly at one point, the error is though within just few percent. Therefore, the model in its simplified form, does not specify a unique unification energy, it however leads to the correct representations of the fermions with respect to the gauge group of the SM, the Higgs doublet appears as part of the inner fluctuations of the metric, and Spontaneous Symmetry Breaking mechanism arises naturally with the negative mass term without any tuning. Moreover, the see-saw mechanism is obtained, the 16 fundamental fermions are recovered, and a top quark mass of $M_{\text{top}} \sim 179$ GeV is predicted. The mass of Higgs in zeroth order approximation of the spectral action is ~ 170 GeV, which is strictly speaking ruled out by current experimental data. Nevertheless, the result depends on the value of gauge couplings at unification scale, which is certainly uncertain, while it was found neglecting the nonminimal coupling between the Higgs field and the Ricci curvature. It is however worth noticing that the NCG approach leads to the correct order of magnitude for the Higgs mass, a result which was not obvious *a priori*.

3. Cosmological consequences

To use the formalism of spectral triples in NCG, it is convenient to work with Euclidean rather than Lorentzian signature. Thus, the analysis of the cosmological consequences of the theory relies on a Wick rotation to imaginary time, into the Lorentzian signature.

The Lorentzian version of the gravitational part of the asymptotic formula for the bosonic sector of the NCG spectral action, including the coupling between the Higgs field and the Ricci curvature scalar, reads [3]

$$\mathcal{S}_{\text{grav}}^{\text{L}} = \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tau_0 R^* R^* \xi_0 R |\mathbf{H}|^2 \right) \sqrt{-g} d^4x , \tag{8}$$

leading to the equations of motion [5]

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \frac{1}{\beta^2}\delta_{cc}\left[2C_{;\lambda;\kappa}^{\mu\lambda\nu\kappa} + C^{\mu\lambda\nu\kappa}R_{\lambda\kappa}\right] = \kappa_0^2\delta_{cc}T_{\text{matter}}^{\mu\nu},$$

with

$$\beta^2 \equiv -\frac{1}{4\kappa_0^2\alpha_0} \quad \text{and} \quad \delta_{cc} \equiv [1 - 2\kappa_0^2\xi_0\mathbf{H}^2]^{-1}. \quad (9)$$

In the low energy weak curvature regime, the nonminimal coupling between the background geometry and the Higgs field can be neglected, implying $\delta_{cc} = 1$. For a Friedmann-Lemaître-Robertson-Walker (FLRW) space-time, the Weyl tensor vanishes, hence the NCG corrections to the Einstein equation vanish [5], rendering difficult to restrict β (or equivalently α_0 , or f_0) via cosmology or solar-system tests. Imposing however a lower limit on β is very important, since it implies an upper limit to the moment f_0 , corresponding to a restriction on the particle physics at unification. This has been achieved in Refs. [8, 9], by considering the energy lost to gravitational radiation by orbiting binaries.

Deriving the weak field limit of noncommutative spectral geometry, we have shown [8] that the production and dynamics of gravitational waves are significantly altered and, in particular, the graviton contains a massive mode that alters the energy lost to gravitational radiation, in systems with evolving quadrupole moment. Considering the rate of energy loss from a binary pair of masses m_1, m_2 , in the far field limit, we have shown [8, 9] that the orbital frequency

$$\omega = |\rho|^{-3/2}\sqrt{G(m_1 + m_2)}, \quad (10)$$

where $|\rho|$ stands for the magnitude of their separation vector, has a critical value

$$2\omega_c = \beta c, \quad (11)$$

around which strong deviations from the familiar results of General Relativity are expected. This maximum frequency results from the natural length scale, given by β^{-1} , at which noncommutative geometry effects become dominant.

The form of the gravitational radiation from binary systems can be used to constrain β . There are several binary pulsars for which the rate of change of the orbital frequency has been well characterised, and the predictions of General Relativity agree with the data to high accuracy. Thus, one can restrict the parameter β by requiring that the magnitude of deviations from General Relativity be less than this uncertainty. Requiring that $\beta > 2\omega_c/c$, we have found [9]

$$\beta > 7.55 \times 10^{-13} \text{ m}^{-1}. \quad (12)$$

Since the strongest constraint comes from systems with high orbital frequencies, one expects that future observations of rapidly orbiting binaries, relatively close to the Earth, could improve it by many orders of magnitude.

Considering the background equations, the corrections to Einstein's equations can only be apparent at leading order for anisotropic models. Calculating the modified Friedmann equation for the Bianchi type-V model, we have shown [5] that the correction terms come in two types. The first one contains terms which are fourth order in time derivatives, hence for the slowly varying functions usually used in cosmology they can be neglected. The second one occurs at the same order as the standard Einstein-Hilbert terms, however, it vanishes for homogeneous versions of Bianchi type-V. Thus, although anisotropic cosmologies do contain corrections due to the additional NCG terms in the action, they are typically of higher order. Inhomogeneous models do

contain correction terms that appear on the same footing as the ordinary (commutative) terms. In conclusion, the corrections to Einstein's equations can only be important for inhomogeneous and anisotropic space-times.

Certainly one cannot always neglect the coupling of the Higgs field to the curvature. Namely, as energies approach the Higgs scale, this nonminimal coupling can no longer be neglected, leading to corrections even for background cosmologies. To understand the effects of these corrections let us neglect the conformal term in Eq. (9), so that the equations of motion read [5]

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa_0^2 \left[\frac{1}{1 - \kappa_0^2|\mathbf{H}|^2/6} \right] T_{\text{matter}}^{\mu\nu} . \quad (13)$$

Thus, $|\mathbf{H}|$ plays the rôle of an effective gravitational constant [5].

The nonminimal coupling between the Higgs field and the Ricci curvature may turn out to be particularly useful in early universe cosmology [6, 7]. Such a coupling has been introduced *ad hoc* in the literature, in an attempt to drive inflation through the Higgs field. However, the value of the coupling constant between the scalar field and the background geometry should be dictated by the underlying theory.

In a FLRW metric, the Gravity-Higgs sector of the asymptotic expansion of the spectral action, in Lorentzian signature, reads

$$S_{\text{GH}}^{\text{L}} = \int \left[\frac{1 - 2\kappa_0^2\xi_0 H^2}{2\kappa_0^2} R - \frac{1}{2}(\nabla H)^2 - V(H) \right] \sqrt{-g} d^4x , \quad (14)$$

where

$$V(H) = \lambda_0 H^4 - \mu_0^2 H^2 , \quad (15)$$

with μ_0 and λ_0 subject to radiative corrections as functions of energy. For large enough values of the Higgs field, the renormalised value of these parameters must be calculated, while the running of the top Yukawa coupling and the gauge couplings must be evolved simultaneously.

At high energies the mass term is sub-dominant, and can be neglected. For each value of the top quark mass, there is a value of the Higgs mass where the effective potential is about to develop a metastable minimum at large values of the Higgs field and the Higgs potential is locally flattened [7]. Since the region where the potential is flat is narrow, the slow-roll must be very slow in order to provide a sufficiently long period of quasi-exponential expansion. Besides the slow-roll parameters, denoted by ϵ and η , which may be slow enough to allow sufficient number of e-folds, the amplitude of density perturbations $\Delta_{\mathcal{R}}^2$ in the Cosmic Microwave Background must be within the allowed experimental window. Inflation predicts that at horizon crossing (denoted by stars), the amplitude of density perturbations is related to the inflaton potential through

$$\left(\frac{V_*}{\epsilon_*} \right)^{\frac{1}{4}} = 2\sqrt{3\pi} m_{\text{Pl}} \Delta_{\mathcal{R}}^{\frac{1}{2}} , \quad (16)$$

where $\epsilon_* \leq 1$. Its value, as measured by WMAP7 [14], requires

$$\left(\frac{V_*}{\epsilon_*} \right)^{\frac{1}{4}} = (2.75 \pm 0.30) \times 10^{-2} m_{\text{Pl}} , \quad (17)$$

where m_{Pl} stands for the Planck mass.

Calculating [7] the renormalisation of the Higgs self-coupling up to two-loops, we have constructed an effective potential which fits the renormalisation group improved potential around

the flat region. We have found [7] a very good analytic fit to the Higgs potential around the minimum of the potential:

$$V^{\text{eff}} = \lambda_0^{\text{eff}}(H)H^4 = [a \ln^2(b\kappa H) + c]H^4 , \quad (18)$$

where the parameters a, b are related to the low energy values of top quark mass m_t as [7]

$$\begin{aligned} a(m_t) &= 4.04704 \times 10^{-3} - 4.41909 \times 10^{-5} \left(\frac{m_t}{\text{GeV}} \right) + 1.24732 \times 10^{-7} \left(\frac{m_t}{\text{GeV}} \right)^2 , \\ b(m_t) &= \exp \left[-0.979261 \left(\frac{m_t}{\text{GeV}} - 172.051 \right) \right] . \end{aligned} \quad (19)$$

The third parameter, c , encodes the appearance of an extremum and depends on the values for top quark mass and Higgs mass. An extremum occurs if and only if $c/a \leq 1/16$, the saturation of the bound corresponding to a perfectly flat region. It is convenient to write $c = [(1+\delta)/16]a$, where $\delta = 0$ saturates the bound below which a local minimum is formed.

This analysis was performed in the case of minimal coupling, so let us investigate the modifications introduced in the case of a small nonminimal coupling; within NCG the coupling is $\xi_0 = 1/12$. We have found [7] that the induced corrections to the potential imply that flatness does not occur at $\delta = 0$, but for fixed values of δ depending on the value of the top quark mass. Thus, for inflation to occur via the Higgs field, the top quark mass fixes the Higgs mass extremely accurately. Scanning carefully through the parameter space, we concluded [7] that sufficient e -folds are indeed generated provided a suitably tuned relationship between the top quark mass and the Higgs mass holds. However, while the Higgs potential can lead to the slow-roll conditions being satisfied once the running of the self-coupling at two-loops is included, the constraints imposed from the CMB data make the predictions of such a scenario incompatible with the measured value of the top quark mass [7]. Running of the gravitational constant and corrections by considering the more appropriate de Sitter, instead of a Minkowski, background we found [7] that do not improve substantially the realisation of a successful inflationary era.

4. Conclusions

Noncommutative spectral geometry is a beautiful mathematical construction with a rich phenomenological arena. In its present *simple* form, which still remains classical and refers only to almost commutative spaces, it offers an elegant explanation for the Standard Model of electroweak and strong interactions. The prediction for the top quark mass is in agreement with current experimental data, while the Higgs mass is within the correct order of magnitude; its precise value is excluded from the most recent experimental data but it is still remarkable how close it remains to the experimental allowed value besides the simplifications under which it was calculated.

Noncommutative spectral geometry lives by construction at very high energy scales, offering a natural set-up to study early universe cosmology. Late time astrophysics is a more difficult task due to technical issues at the current stage of the NCG spectral action [4]. Expecting further progress in computing exactly the spectral action in its nonperturbative form and performing the appropriate renormalisation group analysis, we expect that we will be able to tackle astrophysical issues in the near future.

Here, I have reviewed possible cosmological fingerprints of noncommutative spectral geometry, and proposed a mechanism to constrain physics at unification through the implications of production and dynamics of gravitational waves.

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